

On super Catalan polynomials

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Abstract

We present two q -analogs of the super Catalan numbers, which also generalize Carlitz's q -Catalan numbers $c_n(\lambda)$ for $\lambda = 0$ and $\lambda = 1$. We give a combinatorial interpretation for one of these analogs when $m = 2$.

1 Introduction

In [6] Gessel reintroduces the integers

$$S(m, n) = \frac{\binom{2m}{m} \binom{2n}{n}}{\binom{m+n}{n}} = \frac{(2m)!(2n)!}{m!n!(m+n)!},$$

which were studied by Eugene Catalan in 1874 [4]. When $m, n > 0$, the numbers $S(m, n)$ are even. Gessel refers to $T(m, n) = S(m, n)/2$ as the super Catalan numbers. An interpretation of $T(2, n)$ in terms of pairs of Dyck paths with restricted heights has been found by Gessel and Xin [7].

We will use the standard q -notation

$$[r]_q = 1 + q + \cdots + q^{r-1} \quad \text{and} \quad [n]!_q = \prod_{r=1}^n [r]_q.$$

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The polynomials

$$S_q(m, n) = \frac{[2m]!_q [2n]!_q}{[m]!_q [n]!_q [m+n]!_q}$$

have been studied by Warnaar and Zudilin [8], and by Allen [1]. Warnaar and Zudilin proved that $S_q(m, n)$ are polynomials with nonnegative integer coefficients [8]. Allen conjectured that $S_q(m, n)$ are unimodal [1].

Let $\mathfrak{S}(m, n)$ denote the set of lattice paths that begin at the origin, have m *up* steps (drawn by ascending edges) and n *down* steps (drawn by descending edges). Let $\mathfrak{S}_+(m, n)$ denote the subset of those paths which never go below the x -axis. We will refer to $\mathfrak{S}_+(n, n) = \mathcal{C}_n$ as the set of Catalan paths of length $2n$. The height of $\pi \in \mathfrak{S}(m, n)$, which is the maximum level π reaches, will be denoted by $h(\pi)$. A path of length ℓ can be represented as a sequence of zeros and ones, $\pi = \pi_1 \cdots \pi_\ell$, where zeros represent *up* steps and ones represent *down* steps.

For a lattice path π we define the descent set $D(\pi)$, the major index $maj(\pi)$, and the descent index $des(\pi)$ to be

$$\begin{aligned} D(\pi) &= \{i : \pi_i > \pi_{i+1}, 1 \leq i \leq \ell - 1\}, \\ maj(\pi) &= \sum_{i \in D(\pi)} i, \\ des(\pi) &= |D(\pi)|. \end{aligned}$$

A combinatorial interpretation of $S_q(0, n)$ is given by Andrews [3]

$$S_q(0, n) = \left[\begin{matrix} 2n \\ n \end{matrix} \right]_q = \sum_{\pi \in \mathfrak{S}(n, n)} q^{maj(\pi)}.$$

Allen showed that $T_q(m, n) = S_q(m, n)/(1 + q^n)$ and $U_q(m, n) = S_q(m, n)/(1 + q^m)$ are polynomials [1]. In fact

$$T_q(1, n) = \sum_{\pi \in \mathfrak{S}_+(n, n)} q^{maj(\pi) - des(\pi)} \quad \text{and} \quad U_q(1, n) = \sum_{\pi \in \mathfrak{S}_+(n, n)} q^{maj(\pi)}$$

are respectively Carlitz's q -Catalan numbers $c_n(0)$ and $c_n(1)$ [5].

The super Catalan numbers satisfy the following identity, attributed to Dan Rubenstein [6]

$$4T(m, n) = T(m+1, n) + T(m, n+1). \tag{1}$$

The following q -analog of this identity holds

$$(1 + q^n)(1 + q^{n-m})T_q(m, n) = q^{n-m}T_q(n, m + 1) + T_q(m, n + 1). \quad (2)$$

In Section 2 we provide several results on q -analog Ballot Numbers. In Section 3 we expand on our methods in [2] to give a combinatorial interpretation of $T_q(2, n)$.

2 A q -analog Ballot Number

Let $\mathcal{B}(n, r)$ denote the set of paths of length $2n$ which begin at the origin with an *up* step, end at $(2n, -2r + 2)$, and never go below the line $y = -2r + 2$. In particular $\mathcal{B}(n, 1) = \mathcal{C}_n$, the set of Catalan paths of length $2n$.

Define the q -analog Ballot Number

$$B_q(n, r) = \frac{[2n - 1]_q! [2r]_q}{[n + r]_q! [n - r]_q!} = \frac{1}{q^{n-r}} \left(\begin{bmatrix} 2n-1 \\ n+r-1 \end{bmatrix}_q - \begin{bmatrix} 2n-1 \\ n+r \end{bmatrix}_q \right).$$

Lemma 1. *Let $\pi \in \mathfrak{S}(m, n)$ ending with an up step. Reflecting π over the x -axis gives a path $\rho \in \mathfrak{S}(n, m)$ ending with a down step which satisfies $\text{maj}(\rho) = \text{maj}(\pi) + n$.*

Proof. Given a path $\pi \in \mathfrak{S}(m, n)$ ending with an *up* step, let $D(\pi) = \{X_1, \dots, X_\ell\}$. We let $X_0 = 0$. Define u_i and d_i to be the number of *up* and *down* steps, respectively, between indices X_i and X_{i+1} . Let ρ be the reflection of π across the x -axis. Then the descents of ρ occur exactly at indices $X_i + u_i$ for $i < \ell$. Hence,

$$\text{maj}(\rho) = \sum_{i=0}^{\ell-1} (X_i + u_i) = \text{maj}(\pi) + m - (X_\ell + u_\ell) = \text{maj}(\pi) + m - (n + m) = \text{maj}(\pi) - n.$$

Theorem 1.

$$B_q(n, r) = \sum_{\pi \in \mathcal{B}(n, r)} q^{\text{maj}(\pi) - \text{des}(\pi)} \quad (3)$$

Proof. Let $\mathfrak{S}_>$ denote the set of paths in $\mathfrak{S}(n+r-1, n-r)$ which have height strictly greater than $2r-1$, and let \mathfrak{S}_\leq denote the set of paths in $\mathfrak{S}(n+r-1, n-r)$ which never go above $y = 2r-1$.

We will define a bijection $\psi : \mathfrak{S}_{>} \rightarrow \mathfrak{S}(n+r, n-r-1)$ which preserves the major index. Given a path $\pi \in \mathfrak{S}_{>}$, let R be the right-most highest point on π . Since π has height strictly greater than $2r-1$ and ends at level $2r-1$, the point R is not the last point on π . Let RL be the *down* step following R . Define $\psi(\pi)$ to be the path obtained from π by changing the *down* step RL into an *up* step. Note that $\psi(\pi) \in \mathfrak{S}(n+r, n-r-1)$ and L is the left-most highest point on $\psi(\pi)$. To see that ψ is a bijection from $\mathfrak{S}_{>}$ to $\mathfrak{S}(n+r, n-r-1)$, given a path ρ in $\mathfrak{S}(n+r, n-r-1)$, locate the left-most highest point L on ρ and change the *up* step preceding it into a *down* step to obtain π . Since ρ has height at least $2r+1$, the path π will have height at least $2r$. Therefore $\pi \in \mathfrak{S}_{>}$ and $\psi(\pi) = \rho$. The bijection ψ preserves the major index because the descent sets of π and $\psi(\pi)$ are the same. It follows that

$$\left[\begin{matrix} 2n-1 \\ n+r-1 \end{matrix} \right]_q - \left[\begin{matrix} 2n-1 \\ n+r \end{matrix} \right]_q = \sum_{\pi \in \mathfrak{S}(n+r-1, n-r)} q^{\text{maj}(\pi)} - \sum_{\pi \in \mathfrak{S}(n+r, n-r-1)} q^{\text{maj}(\pi)} = \sum_{\pi \in \mathfrak{S}_{\leq}} q^{\text{maj}(\pi)}.$$

We will define a bijection $\varphi : \mathfrak{S}_{\leq} \rightarrow \mathcal{B}(n, r)$. Let $\pi \in \mathfrak{S}_{\leq}$. Define $\varphi(\pi)$ to be the path obtained from π by reflecting π across the x -axis and then adding an *up* step to the beginning of the path. This is clearly a bijection from \mathfrak{S}_{\leq} to $\mathcal{B}(n, r)$. By Lemma 1, reflecting π across the x -axis causes the major index to decrease by $n-r$. Adding an *up* step to the beginning of the path increases all descents by 1, hence the major index of the reflection of π equals the major minus descent index of $\varphi(\pi)$. It follows that,

$$\frac{1}{q^{n-r}} \left(\left[\begin{matrix} 2n-1 \\ n+r-1 \end{matrix} \right]_q - \left[\begin{matrix} 2n-1 \\ n+r \end{matrix} \right]_q \right) = \sum_{\pi \in \mathfrak{S}_{\leq}} q^{\text{maj}(\pi) - (n-r)} = \sum_{\pi \in \mathcal{B}(n, r)} q^{\text{maj}(\pi) - \text{des}(\pi)}.$$

□

3 Combinatorial Interpretation

The following identity is the q -analog of Eq. 2 in [2] when $m=2$.

$$q^{n-1}T_q(2, n) = (1 + q^2)B_q(n, 1) - B_q(n, 2) \quad (4)$$

For a path $\pi \in \mathcal{C}_n$, let X be the last, from left to right, level one point up to and including

the right-most maximum R on π . Let $h_-(\pi)$ denote the maximum level that the path π reaches from its beginning until and including point X , and $h_+(\pi)$ denote the maximum level that the path π reaches after and including point X . Obviously $h_-(\pi) \leq h_+(\pi) = h(\pi)$. Let Ω_n denote the set of $\pi \in \mathcal{C}_n$ such that $h_+(\pi) \leq h_-(\pi) + 2$.

Theorem 2.

$$T_q(2, n) = q^{n-1} + q^{3-n} \sum_{\pi \in \Omega_n} q^{\text{maj}(\pi) - \text{des}(\pi)}. \quad (5)$$

Proof. By $\mathcal{B}^*(n, 2)$ we denote the set of paths in $\mathcal{B}(n, 2)$ which do not attain level $y = -1$ before their right-most maximum. Let $\mathcal{B}^{**}(n, 2) = \mathcal{B}(n, 2) - \mathcal{B}^*(n, 2)$ and

$$B_q^*(n, 2) = \sum_{\pi \in \mathcal{B}^*(n, 2)} q^{\text{maj}(\pi) - \text{des}(\pi)}; \quad B_q^{**}(n, 2) = \sum_{\pi \in \mathcal{B}^{**}(n, 2)} q^{\text{maj}(\pi) - \text{des}(\pi)}.$$

By Theorem 1 and Eq. 4

$$q^{n-1}T_q(2, n) = (B_q(n, 1) - B_q^*(n, 2)) + (q^2B_q(n, 1) - B_q^{**}(n, 2)).$$

First we compute $B_q(n, 1) - B_q^*(n, 2)$. For $\pi \in \mathcal{B}^*(n, 2)$, let RQ be the *down* step that follows the right-most maximum point R of π . We define $f(\pi)$ to be the path obtained by substituting the *down* step RQ by an *up* step. See Figure 1. Note that $f(\pi) \in \mathcal{C}_n$ and, since at least two *up* steps precede Q on $f(\pi)$, the height of $f(\pi)$ is at least two. Also π and $f(\pi)$ have the same set of descents, thus $\text{des}(\pi) = \text{des}(f(\pi))$ and $\text{maj}(\pi) = \text{maj}(f(\pi))$. It is important to mention that Q is the left-most maximum on $f(\pi)$.

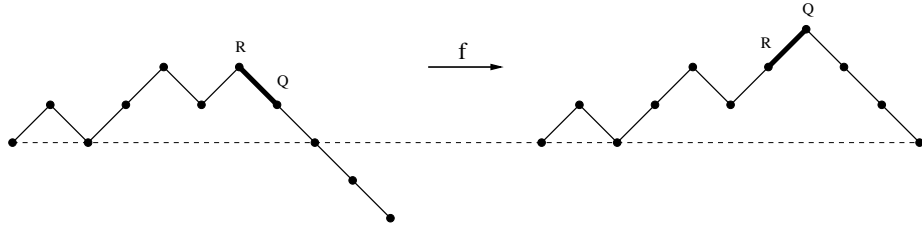


Figure 1: f substitutes the *down* step RQ by an *up* step

We will show that f is a bijection between $\mathcal{B}^*(n, 2)$ and the set of paths ρ in \mathcal{C}_n of height $h(\rho) > 1$. Let Q be the left-most maximum on ρ and RQ be the *up* step that precedes Q .

Substitute the *up* step RQ by a *down* step, which makes R the right-most maximum of the resulting path, call it π . Note that $\pi \in \mathcal{B}^*(n, 2)$ and $f(\pi) = \rho$.

It follows that

$$B_q(n, 1) - B_q^*(n, 2) = \sum_{\substack{\pi \in \mathcal{C}_n \\ h(\pi)=1}} q^{\text{maj}(\pi) - \text{des}(\pi)} = q^{(n-1)^2}.$$

We define a *descent point* to be a point on a path which is preceded by a *down* step, and followed by a *up* step. We define a *down wedge sequence* to be a portion of a path that starts with a *down* step, alternates between *down* steps and *up* steps, and ends with an *up* step. See Figure 2.

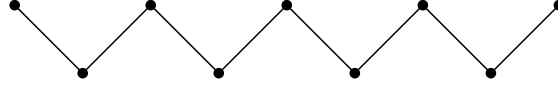


Figure 2: A down wedge sequence

We find a combinatorial interpretation for $q^2 B_q(n, 1) - B_q^{**}(n, 2)$ by establishing an injection g from $\mathcal{B}^{**}(n, 2)$ to \mathcal{C}_n . A path π in $\mathcal{B}^{**}(n, 2)$ attains $y = -1$ before its right-most maximum point R . Let N be the first point before R at which π attains $y = -1$. We consider two cases: when N is immediately followed by a *down* step, and when N is immediately followed by an *up* step.

Case one: N is immediately followed by a *down* step. Since N is the left-most point on π on level $y = -1$, N is preceded by two *down* steps and is followed by one *down* step, then one *up* step. See Figure 3. Let MN be the *down* step that precedes N and NY be the *down* step that follows N . Substitute MY by two *up* steps. The resulting path is a ballot path of length $2n$ that ends at level two. Rename N to be X . From left to right, X is the last level one point on this ballot path. The maximum level that this ballot path reaches up to and including point X is less than the maximum level it reaches after and including point X by at least 4.

Let L be the left-most maximum point of this ballot path and QL be the *up* step that precedes L . Substitute the *up* step QL by a *down* step. See Figure 3. The resulting path $g(\pi)$ is in \mathcal{C}_n and Q is its right-most maximum. Point X is the last level one point on $g(\pi)$ before its right-most maximum Q and the point on $g(\pi)$ before X is a decent. Note that $h_+(g(\pi)) \geq h_-(g(\pi)) + 3$. Also $\text{des}(\pi) = \text{des}(g(\pi))$ and $\text{maj}(\pi) = \text{maj}(g(\pi)) + 2$. Thus $\text{maj}(\pi) - \text{des}(\pi) = \text{maj}(g(\pi)) - \text{des}(g(\pi)) + 2$.

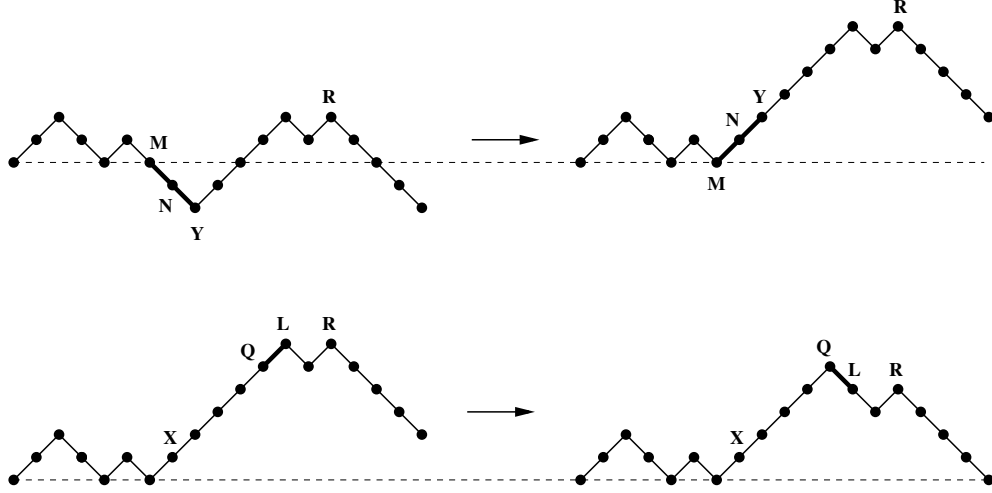


Figure 3: The action of g when N is followed by a *down* step

Case two: N is immediately followed by an *up* step. Since N is the left-most point on π on level $y = -1$, N is preceded by two *down* steps which form a segment we denote by XN . See Figure 4. Let σ be the longest, possibly empty, down wedge sequence that precedes X . Let Y be the first point of the sequence σ . Note that either Y is the second point on π or Y is preceded by a *down* step. Remove σ from its original position and insert it immediately after N . Then substitute XN by two *up* steps. The resulting path is a ballot path of length $2n$ that ends at level two. From left to right, X is the last level one point on this ballot path. The maximum level that this ballot path reaches up to and including point X is less than the maximum level it reaches after and including point X by at least 4.

Let L be the left-most maximum point of this ballot path and QL be the *up* step that precedes L . Substitute the *up* step QL by a *down* step. See Figure 4. The resulting path $g(\pi)$ is in \mathcal{C}_n and Q is its right-most maximum. Note that X is the last level one point on $g(\pi)$ before its right-most maximum Q and the point on $g(\pi)$ before X is NOT a decent. Also $h_+(g(\pi)) \geq h_-(g(\pi)) + 3$. If Y is the second point on the original path π , then g removes the original decent of π that occurs immediately after Y and moves the decent that originally corresponds to N one unit to the left. Thus $des(\pi) = des(g(\pi)) + 1$ and $maj(\pi) = maj(g(\pi)) + 3$. If Y is NOT the second point on the original path π and σ is not empty, then g moves the decent that originally occurs immediately after Y and the one that corresponds to N one unit to the left. If Y is NOT the second point on the original path π and σ is empty, then g moves the decent that corresponds to N two

units to the left. Thus $des(\pi) = des(g(\pi))$ and $maj(\pi) = maj(g(\pi)) + 2$. In all the cases $maj(\pi) - des(\pi) = maj(g(\pi)) - des(g(\pi)) + 2$.

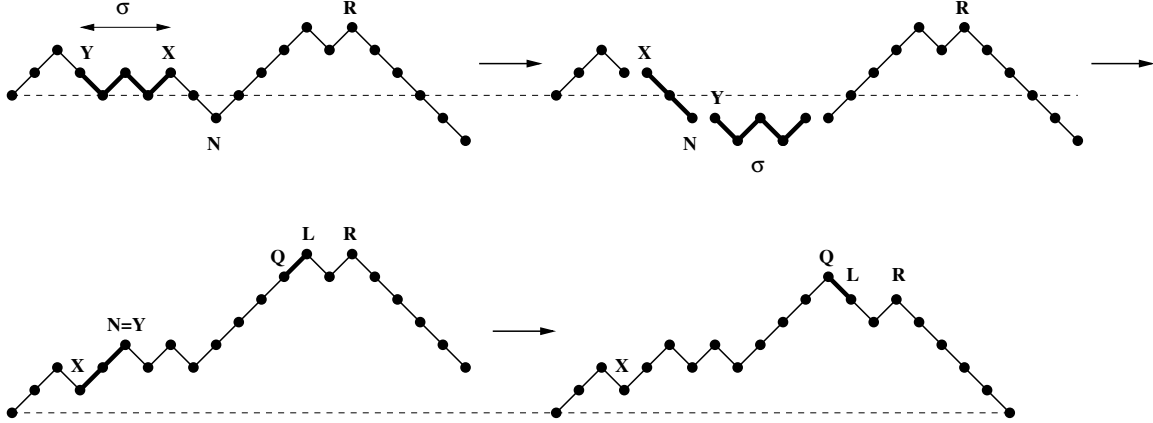


Figure 4: The action of g when N is followed by an *up* step

If a path ρ is in the image of g , then $h_+(\rho) \geq h_-(\rho) + 3$, thus $\rho \in \mathcal{C}_n - \Omega_n$. We will show that the image of g is $\mathcal{C}_n - \Omega_n$. Let ρ be in \mathcal{C}_n and $h_+(\rho) \geq h_-(\rho) + 3$. Let Q be the right-most maximum on ρ and QL be the *down* step that follows Q . Substitute the *down* step QL by an *up* step. The result is a ballot path of length $2n$ that ends at level two. Note that L is the left-most maximum on this ballot path. Let R denote the right-most maximum on this ballot path. From left to right, let X be the last level one point on this ballot path. The maximum level that this ballot path reaches up to and including point X is less than the maximum level it reaches after and including point X by at least 4. We consider two cases: when the point before X is a decent, and when the point before X is not a decent.

If the point before X is a decent, let M be that decent. Let Y be the point that follows X . Since X is the last point on level one before R , XY is an *up* step. Substitute MY with two *down* steps. We will call the resulting path π . Note that π attains level $y = -1$ before its right-most maximum R and, immediately after attaining level $y = -1$ for the first time, it attains level $y = -2$. Thus π is in $\mathcal{B}^{**}(n, 2)$, falls into Case 1, and $g(\pi) = \rho$.

Next we consider the case when the point before X is NOT a decent. Note that, since X is the last level one point before R , X is followed by two *up* steps. Let XY be the segment that consists of these two *up* steps. Let σ be the longest, possibly empty, down wedge sequence that starts at Y . Note that σ is followed by an *up* step. Remove σ from its original position and insert it immediately before X , then substitute XY with two *down* steps. We will call

the resulting path π . Note that π attains level $y = -1$ for the first time at Y , before its right-most maximum R , and Y is followed by an *up* step. Thus π is in $\mathcal{B}^{**}(n, 2)$, falls into Case 2, and $g(\pi) = \rho$.

It follows that

$$q^2 B_q(n, 1) - B_q^{**}(n, 2) = q^2 \sum_{\pi \in \Omega_n} q^{\text{maj}(\pi) - \text{des}(\pi)}.$$

□

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